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THE TEMPERATURE OF A PLATE HEATED BY A SOURCE OF ARBITRARY MOTION AND STRENGTH *

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NOMENCLATURE

a,	width of plate [ft];
b,	thickness of plate [ft];
c_n,	constant;
F,	arbitrary function;
f,	transform of F;
$g(\beta)$,	function [dimensionless];
$h_1, h_2, h_3, h_4, h_5, h_6$,	film coefficients [Btu/h ft^2 °R];
I,	number of subdivisions in x direction;
i,	integer;
J,	number of subdivisions in y direction;
j,	integer;
k,	conductivity [Btu/h ft °R];
l,	length of plate [ft];
m,	integer;
N_{Bi_1}, N_{Bi_2},	Biot number [dimensionless];
n,	integer;
p,	integer;
$\dot{Q}_3$,	distributed source [Btu/h ft^3];
$\dot{q}_3$,	transform of Q_3;
$\dot{Q}_1$,	moving source strength per foot of plate thickness [Btu/h ft];
r,	integer;
t,	time [h];
W,	temperature [°R];
W_s,	temperature of surroundings [°R];
w,	transform of W;
x,	distance [ft];
y,	distance [ft];
z,	distance [ft].

Greek symbols

α,	thermal diffusivity [ft^2/h];
α_1, α_2,	constants;
β,	eigenvalue [dimensionless];
γ,	eigenvalue [ft^{-1}];
∇,	backward difference operator;
θ,	eigenvalue [ft^{-1}];
σ_{1n},	constant [dimensionless];
σ_{2n},	constant [dimensionless];
ϕ,	characteristic function;
$\psi_{m,n}$,	derived constant [h^{-1}].

INTRODUCTION

THE PROBLEM of the temperature distribution in solids due to a moving source is of continuing interest. Theoretical solutions date primarily from the classic paper of Rosenthal [1] who developed the quasi-steady-state theory for a uniform source moving at a uniform velocity in an infinite medium. More recently Cobble [2] treated the problem of a moving discrete source in a finite medium. This paper extends the problem to the case of a continuous source of arbitrary motion and strength, for a thin plate of finite dimensions having the most general Sturm–Liouville boundary conditions.

PROBLEM

The conduction equation for a thin plate, see Fig. 1, having losses on all faces, and having a distributed source, is

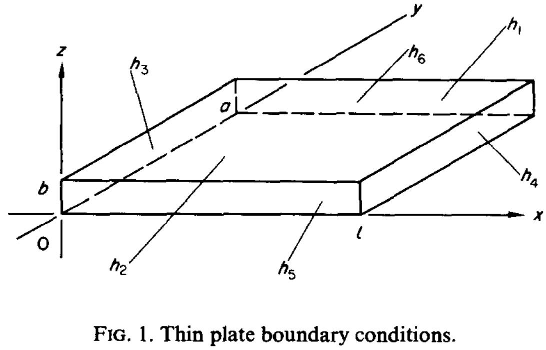

FIG. 1. Thin plate boundary conditions.

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$$\frac{\partial^2 W}{\partial x^2}(x,y,t) + \frac{\partial^2 W}{\partial y^2}(x,y,t) - \frac{(h_1 + h_2)}{kb} W(x,y,t) + \frac{\dot{Q}_3}{k}(x,y,t) = \frac{1}{\alpha}\frac{\partial W}{\partial t}(x,y,t) \qquad (1)$$

where

W, temperature;
h_1, convection coefficient, upper surface;
h_2, convection coefficient, lower surface;
k, conductivity;
b, thickness of plate;
$\dot{Q}_3$, distributed source;
α, thermal diffusivity.

The boundary and initial conditions for a plate having convection losses at the edges, when the surroundings W_s are at zero, are:

1. $\frac{\partial W}{\partial x}(0,y,t) = \frac{h_3}{k} W(0,y,t)$;
2. $\frac{\partial W}{\partial x}(l,y,t) = \frac{-h_4}{k} W(l,y,t)$;
3. $\frac{\partial W}{\partial y}(x,0,t) = \frac{h_5}{k} W(x,0,t)$;
4. $\frac{\partial W}{\partial y}(x,a,t) = \frac{-h_6}{k} W(x,a,t)$;
5. $W(x,y,0) = F(x,y)$.

To solve equation (1) subject to the boundary and initial conditions shown, it is necessary to find the properties of a special linear function $\phi(x)$. Using the methods in [2], the following identities can be developed:

The characteristic functions are

$$\phi_n(x) = \cos\gamma_n x + \frac{N_{Bi_1}}{\beta_n}\sin\gamma_n x, \quad n = 1,2,3,\ldots \qquad (2)$$

where

$$N_{Bi_1} = \frac{h_1 l}{k} \qquad (3)$$

$$\beta_n = \gamma_n l. \qquad (4)$$

The eigenvalue equation is

$$\tan\beta_n = \frac{\sigma_{1n} + \sigma_{2n}}{1 - \sigma_{1n}\sigma_{2n}} \qquad (5)$$

where

$$\sigma_{1n} = \frac{h_1 l}{k\gamma_n l} = \frac{N_{Bi_1}}{\beta_n} \qquad (6)$$

$$\sigma_{2n} = \frac{h_2 l}{k\gamma_n l} = \frac{N_{Bi_2}}{\beta_n}. \qquad (7)$$

The inversion equation is

$$F(x) = \frac{2}{l}\sum_{n=1}^{\infty}\frac{f(\gamma_n)}{g(\beta_n)}\phi_n(x) \qquad (8)$$

where

$$g(\beta_n) = \left\{(1+\sigma_{1n}^2) + \frac{(\sigma_{1n}+\sigma_{2n})[2\sigma_{1n}(\sigma_{1n}+\sigma_{2n}) + (1-\sigma_{1n}^2)(1-\sigma_{1n}\sigma_{2n})]}{\beta_n(1+\sigma_{1n}^2)(1+\sigma_{2n}^2)}\right\}. \qquad (9)$$

A listing of the first fifty eigenvalues for various arguments of N_{Bi_1} and N_{Bi_2}, and $g(\beta_n)$ for the same arguments is given in [3].

The transform of $F(x)$ is

$$T\{F(x)\} = \int_0^l F(x)\phi_n(x)\,dx = f(\gamma_n), \quad n = 1,2,3,\ldots. \qquad (10)$$

The transform of $F''(x)$ is given by

$$T\{F''(x)\} = -\gamma_n^2 f(\gamma_n) + \phi_n(l)\left[F'(l) + \frac{h_2}{k}F(l)\right] - \phi_n(0)\left[F'(0) - \frac{h_1}{k}F(0)\right]. \qquad (11)$$

SOLUTION OF THE TRANSFORM DIFFERENTIAL EQUATION

Taking transforms of equation (1) and the boundary and initial conditions by means of the transform

$$T\{F(x)\} = \int_0^l F(x)\phi_m(\gamma_m x)\,dx = f(\gamma_m), \quad m = 1,2,3,\ldots \qquad (12)$$

where

$$\gamma_m l = \beta_m \qquad (4a)$$

gives rise to the partial differential equation

$$\frac{\partial^2 w}{\partial y^2}(\gamma_m,y,t) - \left[\gamma_m^2 + \frac{(h_1+h_2)}{kb}\right]w(\gamma_m,y,t) + \frac{\dot{q}_3}{k}(\gamma_m,y,t) = \frac{1}{\alpha}\frac{\partial w}{\partial t}(\gamma_m,y,t) \qquad (13)$$

and its accompanying boundary and initial conditions:

1. $\frac{\partial w}{\partial y}(\gamma_m,0,t) = \frac{h_5}{k} w(\gamma_m,0,t)$
2. $\frac{\partial w}{\partial y}(\gamma_m,a,t) = \frac{-h_6}{k} w(\gamma_m,a,t)$
3. $w(\gamma_m,y,0) = f(\gamma_m,y)$

and where

$$\dot{q}_3(\gamma_m, y, t) = \int_0^l \dot{Q}_3(x, y, t)\phi_m(\gamma_m x)\,dx \tag{14}$$

and

$$f(\gamma_m, y) = \int_0^l F(x, y)\phi_m(\gamma_m x)\,dx. \tag{15}$$

Taking transforms of equation (13) and its accompanying initial condition by means of the transform

$$T\{F(y)\} = \int_0^a F(y)\phi_n(\theta_n y)\,dy = f(\theta_n), \quad n = 1, 2, 3, \ldots \tag{16}$$

where

$$\theta_n a = \beta_n \tag{4b}$$

gives rise to the ordinary differential equation

$$\frac{dw}{dt}(\gamma_m, \theta_n, t) + \alpha\left[\gamma_m^2 + \theta_n^2 + \frac{(h_1 + h_2)}{kb}\right] w(\gamma_m, \theta_n, t) = \frac{\alpha}{k}\dot{q}_3(\gamma_m, \theta_n, t) \tag{17}$$

and its initial condition:

1. $w(\gamma_m, \theta_n, 0) = f(\gamma_m, \theta_n)$

where

$$\dot{q}_3(\gamma_m, \theta_n, t) = \int_0^a \dot{q}_3(\gamma_m, y, t)\phi_n(\phi_n y)\,dy \tag{18}$$

and

$$f(\gamma_m, \theta_n) = \int_0^a f(\gamma_m, y)\phi_n(\theta_n y)\,dy. \tag{19}$$

The solution to equation (17) using the initial condition is

$$w(\gamma_m, \theta_n, t) = f(\gamma_m, \theta_n)\exp(-\psi_{m,t}t) + \frac{\alpha}{k}\dot{q}_3(\gamma_m, \theta_n, t)^* \exp(-\psi_{m,n}t) \tag{20}$$

where

$$\psi_{m,n} = \alpha\left[\gamma_m^2 + \theta_n^2 + \frac{(h_1 + h_2)}{kb}\right] \tag{21}$$

and

$$\dot{q}_3(\gamma_m, \theta_n, t)^* \exp(-\psi_{m,n}t) = \int_0^t \dot{q}_3(\gamma_m, \theta_n, \tau)\exp[-\psi_{m,n}(t - \tau)]\,d\tau = \exp(-\psi_{m,n}t)\int_0^t \dot{q}_3(\gamma_m, \theta_n, t)\exp(\psi_{m,n}t)\,dt. \tag{22}$$

Equation (20) can be written

$$w(\gamma_m, \theta_n, t) = w_i(\gamma_m, \theta_n, t) + w_s(\gamma_m, \theta_n, t) \tag{23}$$

where the subscript i refers to the initial condition solution, and the subscript s refers to the source solution.

NATURE OF THE SOURCE

The transformed source term can be written as

$$w_s(\gamma_m, \theta_n, t) = \frac{\alpha}{k}\int_0^t\int_0^a\int_0^l \dot{Q}_3(x, y, t)\,G(x, y, t)\,dx\,dy\,dt. \tag{24}$$

Defining

$$x_i = \nabla x_1 + \nabla x_2 + \ldots + \nabla x_i \tag{25}$$

$$y_j = \nabla y_1 + \nabla y_2 + \ldots + \nabla y_j \tag{26}$$

$$t_p = \nabla t_1 + \nabla t_2 + \ldots + \nabla t_p \tag{27}$$

where ∇ is the backward difference operator. In the limit as $\nabla x_i \to 0$, $\nabla y_i \to 0$ and $\nabla t_p \to 0$, equation (24) can be written as

$$w_s(\gamma_m, \theta_n t) = \lim_{\substack{\nabla x_i \to 0\\ \nabla y_i \to 0\\ \nabla t_i \to 0}} \frac{\alpha}{k}\sum_{p=1}^{P}\sum_{j=1}^{J}\sum_{i=1}^{I} \times Q_3(x_i, y_j, t_p)\,G(x_i, y_j, t_p)\nabla x_i \nabla y_j \nabla t_p. \tag{28}$$

Now the source term $\dot{Q}_3(x, y, t)$ in Btu/h ft^3, can as $\nabla x_i \to 0$, $\nabla y_j \to 0$, be written in terms of the energy rate input per foot of plate thickness absorbed in the little quadrilateral $\nabla x_i \nabla y_j$. This energy rate input per foot is designated as $\dot{Q}_1(x, y, t)$ in Btu/h ft, and so

$$\dot{Q}_3(x_i, y_j, t_p) = \lim_{\substack{\nabla x_i \to 0\\ \nabla y_j \to 0}} \frac{\dot{Q}_1(x_i, y_j, t_p)}{\nabla x_i \nabla y_j}. \tag{29}$$

Using equation (29) in equation (28), we can write

$$w_s(\gamma_m, \theta_n, t) = \lim_{\substack{\nabla x_i \to 0\\ \nabla y_j \to 0\\ \nabla t_p \to 0}} \frac{\alpha}{k}\sum_{p=1}^{P}\sum_{k=1}^{K}\sum_{i=1}^{I} \times Q_1(x_i, y_j, t_p)\nabla t_p\,G(x_i, y_j, t_p). \tag{30}$$

From Fig. 2, it is evident that when $p \neq j \neq i$, $\dot{Q}_1(x_i, y_j, t_p) = 0$, and also $\dot{Q}_1(x_i, y_j, t_p) = 0$ for $i > I$, so

$$w_s(\gamma_m, \theta_n, t) = \lim_{\substack{\nabla x_i \to 0\\ \nabla y_i \to 0\\ \nabla t_i \to 0}} \frac{\alpha}{k}\sum_{i=1}^{I} Q_1(x_i, y_i, t_i)\nabla t_i\,G(x_i, y_i, t_i). \tag{31}$$

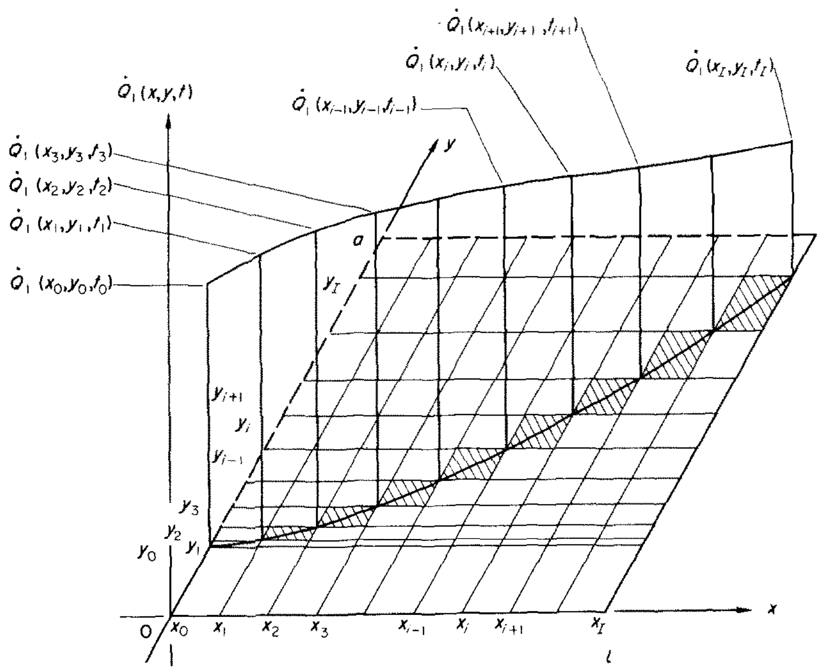

FIG. 2. Moving source on x, y surface.

Thus the transform solution becomes

$$w_s(\gamma_m, \theta_n, t) = f(\gamma_m, \theta_n)\exp(-\psi_{m,n}t) + \lim_{\substack{\nabla x_i \to 0 \\ \nabla y_i \to 0 \\ \nabla t_i \to 0}} \frac{\alpha}{k}\sum_{i=1}^{I} \dot{Q}_1(x_i, y_i, t_i)\nabla t_i \phi_m(\gamma_m x_i)\phi_n(\theta_n y_i) \times \exp[-\psi_{m,n}(t - t_i)]. \quad (32)$$

SOLUTION

Using the inversion equation, the solution can be written as

$$W(x, y, t) = \sum_{m=1}^{\infty} c_m \phi_m(\gamma_m x) = \frac{2}{l}\sum_{m=1}^{\infty} \frac{w(\gamma_m, y, t)}{g(\beta_m)}\phi_m(\gamma_m x) \quad (33)$$

and similarly

$$w(\gamma_m, y, t) = \sum_{n=1}^{\infty} c_n \phi_n(\theta_n y) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{w(\gamma_m, \theta_n, t)}{g(\beta_n)}\phi_n(\theta_n y) \quad (34)$$

and so

$$W(x, y, t) = \frac{4}{al}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{w(\gamma_m, \theta_n, t)\phi_m(\gamma_m x)\phi_n(\theta_n y)}{g(\beta_m)\, g(\beta_n)} \quad (35)$$

in a slightly expanded form

$$W(x, y, t) = \frac{4}{al}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\phi_m(\gamma_m x)\phi_n(\theta_n y)}{g(\beta_m)\, g(\beta_n)} \times \Bigg\{\exp(-\psi_{m,n}t)\int_0^a\int_0^l F(x, y)\phi_m(\gamma_m x)\phi_n(\theta_n y)\,\mathrm{d}x\,\mathrm{d}y + \lim_{\substack{\nabla y_i \to 0 \\ \nabla x_i \to 0 \\ \nabla t_i = 0}} \frac{\alpha}{k}\sum_{i=1}^{I} Q_1(x_i, y_i, t_i)\nabla t_i \phi_m(\gamma_m x_i \phi_n(\theta_n y_i) \times \exp[-\psi_{m,n}(t - t_i)]\Bigg\} \quad (36)$$

for $0 \leqslant x \leqslant l$, $0 \leqslant y \leqslant a$, and $t \geqslant t_I$.

For

$$t_r \leqslant t < t_{r+1}, \quad r = 1, 2, 3, \ldots \quad (37)$$

the solution is the same as equation (36) except that the summation on i stops at r instead of I.

In practice evaluation of equation (36) would be accomplished by breaking up the path of the continuous source $\dot{Q}_1$ across the plate into a finite number of time increments as shown in Fig. 2. Using backward differences, the source strength released in the first shaded area $\nabla x_1 \nabla y_1$ would be evaluated as $\dot{Q}_1(x_1, y_1, t_1)$. The source strength released in the second shaded area $\nabla x_2 \nabla y_2$ would be evaluated as $\dot{Q}_1(x_2, y_2, t_2)$. Similar statements hold up to the Ith shaded area $\nabla x_I \nabla y_I$, in which the source strength would be evaluated at $\dot{Q}_1(x_I, y_I, t_I)$. In general, the smaller the increments, the more accurate the answer.

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